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COMMENT

Deterministic model for Eden trees

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Abstract. We propose and study a deterministic fractal model on the square lattice for the Eden trees, recently proposed by Dhar and Ramaswamy. We find the fractal and spectral dimensions of this fractal structure which are equal to 2 and 4 ln 2/ln 6 respectively.

Among different cluster growth processes, the Eden model (Eden 1961) is the simplest. In this process particles are added to the cluster one after another by being placed randomly on the perimeter of the growing aggregate, resulting in compact clusters.

Recently Dhar and Ramaswamy (1985) have introduced and studied Eden trees. Growth of these clusters is similar to that of the Eden processes except for the restriction that those perimeter sites which have more than one particle in the neighbouring positions are excluded. The most interesting fact is that these clusters have loopless structures which means that any two sites can be connected by one and only one path. Though the Hausdorff or fractal dimensions D of these clusters are equal to their embedding Euclidean space dimensions (d), these clusters are associated with non-trivial spectral dimensions (d_s). Using a generalised node counting theorem and by simulating random walks on these clusters it has been shown that $d_s = 1.22$ for $d = 2$ (Dhar and Ramaswamy 1985). Very recently Havlin *et al* (1985) have calculated the intrinsic dimensions (d_i) of different treelike structures.

Here we propose and study a deterministic model for Eden trees on the square lattice. The cluster grows in this case by successive stages. At the zeroth stage the cluster consists of one particle, i.e. the seed. To grow the cluster from the sth stage to the $(s+1)$th stage we first find out all the perimeter sites of this sth stage cluster and then out of these positions we exclude those sites which have more than one particle in the neighbouring positions. Particles are then placed at all these allowed perimeter sites. In figure 1, the first column shows this cluster at different stages of growth. These clusters are self-similar in different stages of growth, e.g. $s = 1, 3, 7, 15, \ldots$, etc. We specify these self-similar clusters by the order of cluster growth n ($n = 1, 2, 3, \ldots$, for $s = 1, 3, 7, 15, \ldots$, etc). This cluster is finitely ramified in the sense that, to isolate any number of sites connected to a particular site, one would have to cut a finite number of bonds.

We measure the size of these clusters by the maximum of the Euclidean distances between any pair of points on the clusters and denote it by l. Sizes of clusters at different orders of growth follow the recursion relation

$$l_n = 2l_{n-1} + 2. \tag{1}$$

The number of particles at the nth order of the cluster (N_n) is given by the relation

$$N_n = 4N_{n-1} + 1. \tag{2}$$

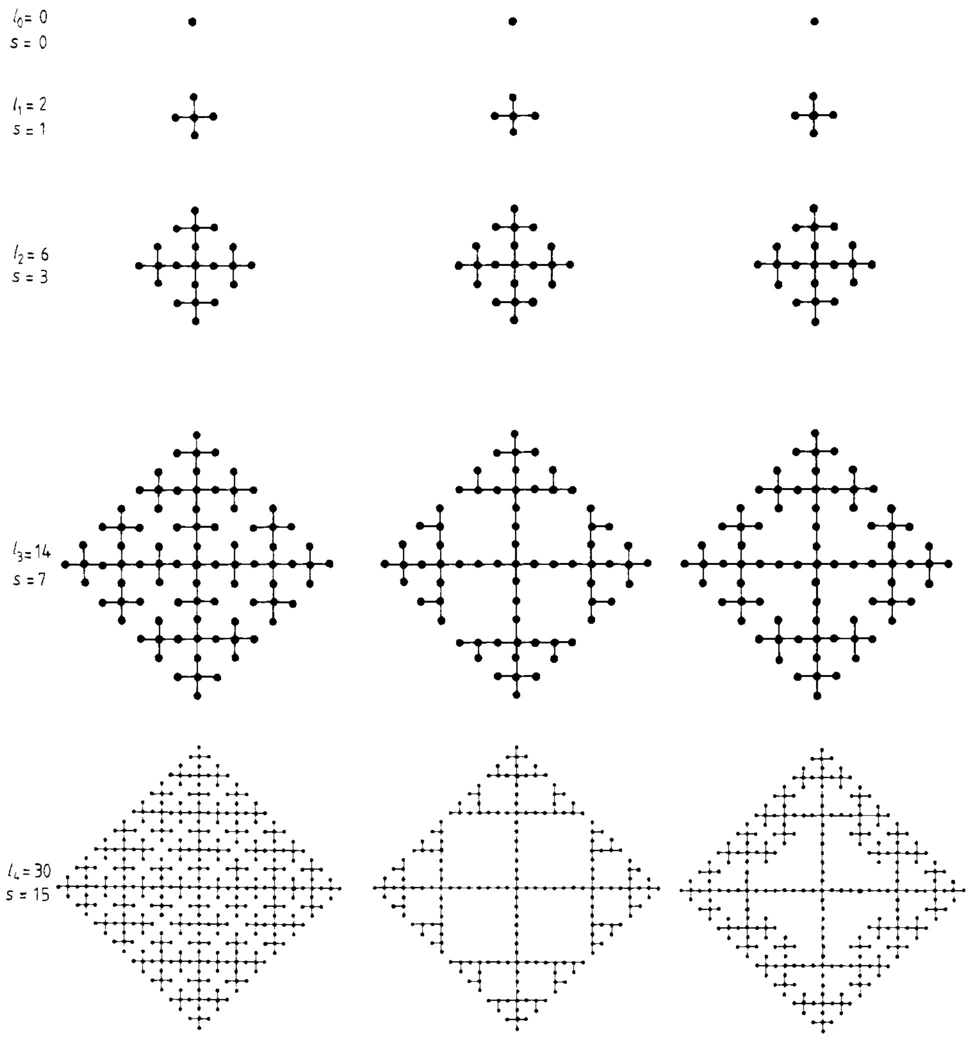

Figure 1. Figure of deterministic Eden trees for different orders of growth (subscripts of l denote the order). The first column shows the cluster, the second column shows the backbone and the third column shows the skeleton.

The fractal dimension of these clusters is

$$D = \lim_{n \to \infty} \ln(N_n / N_{n-1}) / \ln(l_n / l_{n-1}) = 2.$$

Therefore the fractal dimension of the cluster is equal to the embedding Euclidean space dimension.

The backbone (containing sites connecting the seed to the perimeter) at different orders of cluster growth is shown in the second column of figure 1. We find a relation for the number of particles N_n^{B} on the backbone of order n to be

$$N_n^{\mathrm{B}} = 2N_{n-1}^{\mathrm{B}} + 2l_n - 1. \tag{3}$$

This relation shows that the fractal dimension D^{B} of the backbone is equal to 1.

The chemical distance between two points on the cluster is the shortest path of occupied sites linking these two points (Havlin *et al* 1984). As our cluster is loopless, any two points on it are connected by the chemical path only. The chemical distance of any site from the seed site is equal to the s value of that site. The intrinsic dimension (d_i) is defined by the relation $N_n \sim s_n^{d_i}$, where N_n is the number of those sites whose chemical distances from the seed site are less than or equal to s_n units. As $s_n = l_n/2$, the intrinsic dimension is equal to the fractal dimension of the cluster and d_i is equal to 2 for our model.

The skeleton of this cluster consists of the set of sites lying on the chemical paths connecting the seed to those sites which are at a chemical distance equal to s from the seed (Havlin *et al* 1984). Skeletons of our clusters at different orders of growth are shown in the third column of figure 1. The number of particles N_n^s on the nth order skeleton is given by the following recursion relation:

$$N_n^s = 3N_{n-1}^s + l_n. \tag{4}$$

Using this relation we find the fractal dimension of the skeleton to be $D^s = \ln 3/\ln 2$.

On the skeleton for chemical distance s of this cluster the number of sites within chemical distance s' (where $s' \ll s$) is given by $N_n^{s'} \sim s'^{d_i^s}$, where d_i^s is the intrinsic dimension of the skeleton (Havlin *et al* 1985). From the figure showing the skeletons we see that if the cluster is grown up to the stage s then up to the chemical distance $s/2$ the cluster is linear. This means the intrinsic dimension d_i^s of the skeleton is 1.

Now we find the spectral dimension (d_s) of these clusters. If the system is made of equal mass points connected by springs of equal force constants, then the low energy density of states $\rho(\omega)$ for frequency ω depends on spectral dimension as

$$\rho(\omega) \sim \omega^{d_s - 1}. \tag{5}$$

To find out d_s, we follow the decimation procedure of Rammal and Toulouse (1983). Using their notation we write down equations for λ ($=\omega^2/\omega_0^2$, where ω_0 is the microscopic frequency) for the transformation (with scale factor $b = 2$, as shown in figure 2) as follows:

$$\begin{aligned}
(4-\lambda)X_S &= x_1 + x_2 + x_3 + x_4 \\
(2-\lambda)x_1 &= X_1 + X_S \\
(2-\lambda)x_2 &= X_2 + X_S \\
(2-\lambda)x_3 &= X_3 + X_S \\
(2-\lambda)x_4 &= X_4 + X_S \\
(4-\lambda')X_S &= X_1 + X_2 + X_3 + X_4.
\end{aligned} \tag{6}$$

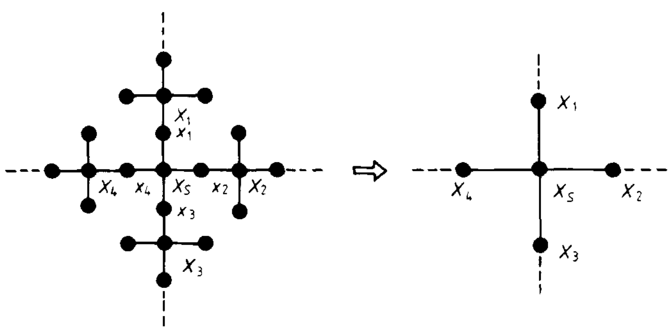

Figure 2. Decimation of the deterministic Eden tree to its next self-similar form with scale factor $b = 2$.

These equations give

$$\lambda' = \lambda(6-\lambda). \tag{7}$$

Using this relation we obtain the spectral dimension of the cluster $d_s = 4\ln 2/\ln 6$.

At this stage we would like to recall the paper of Dhar (1977) where the idea of spectral dimension was first introduced. In this paper Dhar defined the truncated four-simplex lattice. Our cluster has much in common with this lattice, with the difference that to obtain the lattice from our cluster we have to delete some sites and add some bonds from our cluster. However we see that in our case, this difference in the number of sites and bonds does not affect the fractal and spectral dimensionalities.

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